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Recognizing edge clique graphs among interval graphs and probe interval graphs[☆]

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Abstract

The edge clique graph of a graph H is the one having the edge set of H as vertex set, two vertices being adjacent if and only if the corresponding edges belong to a common complete subgraph of H. We characterize the graph classes {edge clique graphs} $\cap$ {interval graphs} as well as {edge clique graphs} $\cap$ {probe interval graphs}, which leads to polynomial time recognition algorithms for them. This work generalizes corresponding results in [M.R. Cerioli, J.L. Szwarcfiter, Edge clique graphs and some classes of chordal graphs, Discrete Math. 242 (2002) 31–39].

Keywords: Edge clique graphs; Interval graphs; Probe interval graphs

1. Edge clique graph

We consider only finite undirected graphs without parallel edges or loops. Let G be a graph. A *clique* of G is a subset of $V(G)$ which induces a complete subgraph in G. We denote by $\mathcal{C}(G)$ the set of all maximal cliques of G. For $S \subseteq V(G)$, let $G(S)$ denote the subgraph of G induced by S.

The *edge clique graph* of a graph G, denoted $K_e(G)$, is the one whose vertices are the edges of G and two vertices are adjacent if and only as edges in G their endpoints all belong to a common clique of G. The construction of edge clique graphs is first implicitly used by Kou, Stockmeyer and Wong in 1978 [1], while this concept is first formally introduced by Albertson and Collins in 1984 [2]. Many results and applications of edge clique graphs can be found in [1–12].

The following is a very useful basic result on edge clique graphs.

Proposition 1 (*Albertson and Collins [2]*). *Let H be a graph. There exists a one-to-one correspondence between nontrivial maximal cliques (intersection of nontrivial maximal cliques) of H and maximal cliques (intersection of maximal cliques) of $K_e(H)$. Moreover, if C is a nontrivial maximal clique (intersection of maximal cliques) of H,*

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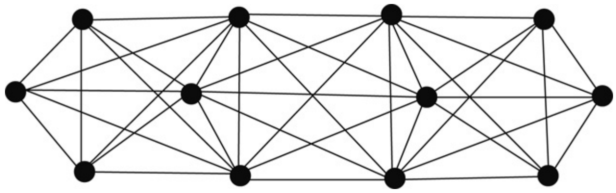

Fig. 1. A graph which satisfies A1 but is not an edge clique graph.

then the corresponding clique of $K_e(H)$ is formed by the vertices which correspond to the edges of H with both endpoints in C.

A *triangular number* is a number of the form $\frac{n(n-1)}{2}$ for some nonnegative integer n. A *triangular clique* is a clique whose size is a triangular number. By Proposition 1 we get

Proposition 2 (*Chartrand et al. [7]*). *Each edge clique graph must satisfy:*

(A1) *any intersection of a set of maximal cliques is a triangular clique.*

A *starlike threshold graph* [6] is a graph which admits an ordering of its maximal cliques as $C_1, \ldots, C_s, C$ so that $C_i \setminus C$ are pairwise disjoint, $C \cap C_i \subseteq C \cap C_{i+1}$ and all vertices coming from the same $C_i \setminus C$ have the same set of closed neighborhood.

Example 3. Cerioli and Szwarcfiter [6] show that for a starlike-threshold graph G, (A1) is a necessary and sufficient condition for it to be an edge clique graph. In general, Chartrand et al. [7] point out that there exists a graph which fulfils (A1) but is not an edge clique graph; see the graph depicted in Fig. 1.

Two characterizations of edge clique graphs have been presented in [5,7], respectively. But there is not yet any polynomial time recognition algorithm for edge clique graphs. As a generalization of the observations made in Example 3, we will introduce interval graphs and probe interval graphs in the next section and then give in Section 3 a polynomial time recognition algorithm for edge clique graphs among these two classes of graphs.

2. Interval graph and probe interval graph

A graph G is an *interval graph* [13,14] if there is a surjective map f from $V(G)$ to a collection S of closed intervals of the real line such that any two different vertices u and v of G are adjacent if and only if $f(u)$ and $f(v)$ have a nonempty intersection. In this case, we also call G the *intersection graph* of (S, f), or simply S. Benzer [15] and Hajös [16] independently initiated the study of interval graphs. Since then, interval graphs have become one of the most useful mathematical structures for modelling real world problems [13, p. 181].

Let $\pi = (C_1, \ldots, C_s)$ be an ordering of maximal cliques of G. We call the linear ordering π a *consecutive clique arrangement* of G, or a *consecutive ordering* of $\mathcal{C}(G)$, provided for every vertex the maximal cliques containing it occur consecutively in π.

Theorem 4 (*Gilmore and Hoffman [17]*). *A graph G is an interval graph if and only if $\mathcal{C}(G)$ has a consecutive ordering.*

Let G be a graph and $V(G)$ be a disjoint union of P and N. Let S be a set of closed intervals of the real line and f a surjective mapping from $V(G)$ to S. We say that (S, f) is a *probe interval representation* of G with respect to (P, N) provided $uv \in E(G)$ if and only if $f(u) \cap f(v) \neq \emptyset$ and at least one of u, v lies in P. The graph G is a *probe interval graph* with respect to (P, N) whenever it has probe interval representation with respect to (P, N). Probe interval graphs are introduced in physical mapping and sequencing of DNA and have received a wide study [18–23]. Especially, we mention that a polynomial time recognition algorithm for probe interval graphs can be found in [18].

3. Main result

To establish our main characterization results, we have to prepare some results on the so-called inverse problem [6, p. 32] for the edge clique graph operator and interval graphs.

Lemma 5. *A graph is both an interval graph and an edge clique graph if and only if it is the edge clique graph of an interval graph.*

Proof. The backward direction is straightforward from Proposition 1 and Theorem 4. So we turn to the forward implication.

Suppose $G = Ke(H)$ is an interval graph. Then, Theorem 4 says that $\mathcal{C}(G)$ has a consecutive ordering $\pi = (C_1, \ldots, C_s)$. Without loss of generality, we may assume that H has no isolated vertex, namely all its maximal cliques are nontrivial. By Proposition 1, each $C_i \in \mathcal{C}(G)$ corresponds to a $Q_i \in \mathcal{C}(H)$ and these Q_i's enumerate all elements of $\mathcal{C}(H)$. If H is itself not an interval graph, then we can locate a vertex $v \in V(H)$ such that the maximal cliques containing v appear in $t_v = t > 1$ segments of consecutive cliques, say cliques among $\cup_{i=1}^{t} S_i$, where $S_i = \{Q_{k_i}, \ldots, Q_{k_i+\ell_i}\}$, $1 \le k_1 < k_1+\ell_1+1 < k_2 < k_2+\ell_2+1 < \cdots < k_t < k_t+\ell_t+1 \le s+1$. It is not hard to see that we can choose t new vertices $v_1, \ldots, v_t$ and construct a new graph H' such that $V(H') = (V(H) \setminus \{v\}) \cup \{v_1, \ldots, v_t\}$ and $E(H') = (E(H) \setminus \{vw : vw \in E(H)\}) \cup (\cup_{i=1}^{t}\{v_i w : vw \in E(H), v, w \in \cup_{Q \in S_i} Q\})$.

Under the most obvious correspondence between $E(H)$ and $E(H')$, we may identify $V(Ke(H))$ with $V(Ke(H'))$. We proceed to show that $E(Ke(H)) = E(Ke(H'))$, which will lead to the conclusion that the above vertex splitting operation does not affect the edge clique graph, that is, $Ke(H') = Ke(H) = G$. This will follow from the fact that

$$\mathcal{C}(H') = \{Q_i' : i = 1, \ldots, s\}, \tag{1}$$

where $Q_i' = (Q_i \setminus \{v\}) \cup \{v_j\}$ if Q_i lies in S_j and $Q_i' = Q_i$ otherwise. To establish Eq. (1), it suffices to check that there is no clique in G which contains vertices v, u, w whenever v, u appear in a clique in S_i and v, w appear in a clique in S_j for $1 \le i < j \le t$. This must be true, because, as π is a consecutive clique arrangement, only those C_m with $m < k_i + \ell_i + 1$ can contain both v and u and only those C_m with $m > k_i + \ell_i + 1$ can contain both v and w.

Clearly, there is one less vertex w with $t_w > 1$ after replacing H by H' and the ordering $(Q_1, \ldots, Q_s)$ by the ordering $(Q_1', \ldots, Q_s')$. Therefore, according to Theorem 4, by continuing this splitting vertex process we will finally come to an interval graph whose edge clique graph is G, finishing the proof. □

An interval graph is said to be *good* provided the size of the intersection of any two different maximal cliques of it does not equal to one.

Lemma 6. *A graph is the edge clique graph of an interval graph if and only if it is the edge clique graph of a good interval graph.*

Proof. Take an interval graph H. Without any loss of generality, let us assume that S is a set of intervals whose endpoints are pairwise distinct and H is the intersection graph of S. Let $a_1 < a_2 < \cdots < a_{2|S|}$ be the set of endpoints of intervals in S. Call a_i a left point if it is the left endpoint of some interval in S and a right point otherwise. A segment $[a_i, a_{i+1}]$ is nice whenever a_i is a left point and a_{i+1} a right point. Due to the Helly property for interval graphs [14, Exercise 2.1.72], we know that the set of nice segments are in one to one correspondence with the set of maximal cliques of H, a nice segment $[a_i, a_{i+1}]$ corresponding to the clique consisting of all vertices whose intervals contain $[a_i, a_{i+1}]$.

The fact that H is not good means that there is an interval $[b, c] \in S$ and two nice segments $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$, $i+1 < j$, such that $[b, c]$ is the unique element from S that covers both $[a_i, a_{i+1}]$ and $[a_j, a_{j+1}]$. Clearly, there is $i + 1 \le k \le j - 1$ satisfying a_k is right and a_{k+1} is left. Replacing $[b, c]$ by two intervals $[b, \frac{2a_k+a_{k+1}}{3}]$ and $[\frac{a_k+2a_{k+1}}{3}, c]$ we get a new set of intervals whose intersection graph shares the same edge clique graph with H. Proceeding with this interval splitting process whenever possible, we will terminate at a good interval graph whose edge clique graph is $Ke(H)$, as wanted. □

Define on the set of triangular numbers a function θ as follows. Let $\theta(0) = 0$ and $\theta(m) = n > 1$ for any positive triangular number $m = \frac{n(n-1)}{2}$. We come to a necessary condition for an interval graph to be an edge clique graph.

Lemma 7. *Let G be an interval graph with a consecutive ordering $\pi = (C_1, \ldots, C_s)$ of $\mathcal{C}(G)$ and put $\alpha_{i,j} = \theta(|C_i \cap C_j|)$ for $s \ge j \ge i \ge 1$. If G is an edge clique graph, then for any $1 \le i \le j \le k \le \ell \le s$ we have*

(A2) $\alpha_{i,k} + \alpha_{j,\ell} \le \alpha_{i,\ell} + \alpha_{j,k}$.

Proof. By Lemmas 5 and 6 we may assume that $G = Ke(H)$ for some good interval graph H, and by Proposition 2 the four terms in Condition (A2) are all well-defined. We only need to show that condition (A2) holds for G. For any $C \in \mathcal{C}(G)$, let C^H be the corresponding element of $\mathcal{C}(H)$, as mentioned in Proposition 1. Since H is good, it holds for any $p \leq q$ that $\alpha_{p,q} = |C_p^H \cap C_q^H|$ and hence (A2) becomes

$$|C_i^H \cap C_k^H| + |C_j^H \cap C_\ell^H| \leq |C_i^H \cap C_\ell^H| + |C_j^H \cap C_k^H|. \tag{2}$$

Furthermore, from the proofs of Lemmas 5 and 6, we know that we can require that $\pi^H = (C_1^H, \ldots, C_s^H)$ is a consecutive ordering of $\mathcal{C}(H)$. For $v \in V(H)$ and a clique Q of H, let $I_v(Q) = 1$ if $v \in Q$ and let $I_v(Q) = 0$ otherwise. By now, to prove (A2), namely Eq. (2), it is enough to show that if v occurs consecutively in an interval, say $\mathcal{F}$, among the ordering of four cliques (A, B, C, D), then

$$I_v(A \cap C) + I_v(B \cap D) \leq I_v(A \cap D) + I_v(B \cap C). \tag{3}$$

Note that, when $\mathcal{F} = (A, B, C)$ or (B, C, D), Eq. (3) is just $1 = 1$; when $\mathcal{F} = (A, B, C, D)$, Eq. (3) becomes $2 = 2$; when $\mathcal{F} = (B, C)$, Eq. (3) turns out to be $0 \leq 1$; and in all other cases, Eq. (3) is simply the trivial relation $0 = 0$. This completes the proof of the lemma. □

From Theorem 4, we can easily check that the graphs considered in Example 3 are all interval graphs. The ensuing theorem says that (A1) together with (A2) is a necessary and sufficient condition for an interval graph to be an edge clique graph, hence providing an easy understanding of the observations in Example 3.

Theorem 8. *Let G be an interval graph with a consecutive ordering $\pi = (C_1, \ldots, C_s)$ of $\mathcal{C}(G)$. Then G is an edge clique graph if and only if, keeping the notation of Lemma 7, it satisfies the following conditions:*

(A1′) *the intersection of any two not necessarily distinct maximal cliques is a triangular clique;*
(A2′) *if $j < \ell$, $C_j \cap C_\ell \neq \emptyset$, then $\alpha_{j-1,\ell-1} + \alpha_{j,\ell} \leq \alpha_{j-1,\ell} + \alpha_{j,\ell-1}$.*

(Note that without condition (A1′) *even the notation $\alpha_{i,j}$ will make no sense.)*

Proof. The necessity part follows from Proposition 2 and Lemma 7.

For the reverse direction, we carry out a proof by induction on s. The assertion is trivially true when $s = 1$. Consider now the case of $s > 1$ under the assumption that the result holds for smaller s. Let G' be the graph with vertex set $\cup_{i=1}^{s-1} C_i$ and $uv \in E(G')$ if and only if $u, v \in C_i$ for some $i = 1, \ldots, s-1$. Since π is a consecutive ordering of $\mathcal{C}(G)$, we can find that $\pi' = (C_1, \ldots, C_{s-1})$ is a consecutive ordering of $\mathcal{C}(G')$. Thus, by induction hypothesis, there is a graph H' such that $G' = Ke(H')$ and the maximal cliques of H' are C_i^H, $i = 1, \ldots, s-1$, where C_i^H corresponds to $C_i \in \mathcal{C}(G)$ in the sense of Proposition 1. From the proof of Lemma 5, we assume that $(C_1^H, \ldots, C_{s-1}^H)$ is a consecutive ordering of $\mathcal{C}(H')$.

Let $i = \min\{t : C_t \cap C_s \neq \emptyset\}$. From (A2′) and that π is consecutive, we deduce that

$$\alpha_{j,s} - \alpha_{j-1,s} \leq \alpha_{j,s-1} - \alpha_{j-1,s-1}, \quad j = i, i+1, \ldots, s-1. \tag{4}$$

Consequently, we can take $\alpha_{j,s} - \alpha_{j-1,s}$ vertices from $(C_j^H \setminus C_{j-1}^H) \cap C_{s-1}^H$, $j = i, i+1, \ldots, s-1$. Denote the union of these vertices by O. Let H be the graph obtained from H' by adding a set N of $\alpha_{s,s} - \alpha_{s-1,s} > 0$ new vertices and adding all edges among these vertices and all edges between these vertices and those in O. Because of Eq. (4), we conclude that $\mathcal{C}(H) = \{C_1^H, \ldots, C_{s-1}^H, C_s^H\}$, where $C_s^H = O \cup N$. In addition, owing to the fact that $(C_1^H, \ldots, C_{s-1}^H)$ is a consecutive ordering of $\mathcal{C}(H')$, we can check that $|C_s^H \cap C_t^H| = \alpha_{t,s}$ for any $t < s$ and then verify that $G = Ke(H)$, ending the proof. □

Theorem 9. *Let G be an edge clique graph. Then G is a probe interval graph if and only if G is an interval graph.*

Proof. It is trivial that every interval graph is a probe interval graph.

To go the other way, let the given edge clique graph G be a probe interval graph with respect to (P, N). There is no loss of generality in assuming that G has no isolated vertices. In view of Proposition 2, this excludes the possibility that there exist $C_1, C_2 \in \mathcal{C}(G)$ such that $|C_1 \setminus C_2| = 1$. Henceforth, with a little thought, it is not hard to argue that

(B) for any $C_1, C_2 \in \mathcal{C}(G)$ which contain $v_1, v_2 \in N$, respectively, $C_1 \cap P$ and $C_2 \cap P$ are different members of $\mathcal{C}(G(P))$.

Take a probe interval representation (S, f) of G with respect to (P, N). We may assume that the endpoints of the intervals in S are pairwise distinct. Let $a_1 < a_2 < \cdots < a_{2|P|}$ be the set of endpoints of those intervals corresponding to the vertices of P. Since $G(P)$ is an interval graph, as shown in the proof of Lemma 6 we know that the set of nice segments with respect to the interval representation $\{f(v) : v \in P\}$ of $G(P)$ are in one to one correspondence with $\mathcal{C}(G(P))$, a nice segment $[a_i, a_{i+1}]$ corresponding to the clique consisting of all vertices from P whose intervals contain $[a_i, a_{i+1}]$.

For any $v \in N$, define $\mathcal{S}_v$ to be the set of nice segments with respect to the interval representation $\{f(u) : u \in P\}$ which have nonempty intersection with $f(v)$. Let $T_v = \cup_{I \in \mathcal{S}_v} I$, $L_v = \min\{x \in T_v\}$ and $R_v = \max\{x \in T_v\}$. By the preceding claim (B), it follows that for $u \neq v \in N$, $T_u \cap T_v = \emptyset$. But both $f(u)$ and $f(v)$ are intervals, which implies that $[L_v, R_v] \cap [L_u, R_u] = \emptyset$.

We are ready to establish an interval representation for G. We just associate with each $v \in P$ the interval $f(v)$ and associate with each $v \in P$ the interval $[L_v, R_v]$. It is a simple matter that the existence of these intervals gives rise to what we want. $\square$

For any graph G, there is an $O(|V(G)|^2)$ time algorithm which either produces a consecutive ordering of $\mathcal{C}(G)$, hence indicating that G is an interval graph, or else outputs the answer that no such ordering exists [13, p. 175] [24]. Note that an interval graph G has at most $|V(G)|$ maximal cliques [25]. So, utilizing any consecutive ordering π of $\mathcal{C}(G)$ we can check whether or not G satisfies conditions (A1$'$) and (A2$'$) within $O(|V(G)|^3)$ time. To sum up, Theorems 8 and 9 allow us to assert that $O(|V(G)|^3)$ time is enough to test whether a given interval graph or a given probe interval graph is an edge clique graph.

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